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On relaxed and contraction-proximal point algorithms in hilbert spaces

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Abstract

We consider the relaxed and contraction-proximal point algorithms in Hilbert spaces. Some conditions on the parameters for guaranteeing the convergence of the algorithm are relaxed or removed. As a result, we extend some recent results of Ceng-Wu-Yao and Noor-Yao.

Keywords: maximal monotone operator, proximal point algorithm, firmly nonexpansive operator

1. Introduction

Throughout, H denotes a real Hilbert space and A a multi-valued operator with domain $D(A)$. We know that A is called monotone if $\langle u - v, x - y \rangle \geq 0$, for any $u \in Ax$, $v \in Ay$; maximal monotone if its graph $G(A) = \{(x, y) : x \in D(A), y \in Ax\}$ is not properly contained in the graph of any other monotone operator. Denote by $S = \{x \in D(A) : 0 \in Ax\}$ the zero set and by $J_c = (I + cA)^{-1}$ the resolvent of A . It is well known that J_c is single valued and $D(J_c) = H$ for any $c > 0$.

A fundamental problem of monotone operators is that of finding an element x so that $0 \in Ax$. This problem is essential because it includes many concrete examples, such as convex programming and monotone variational inequalities. A successful and powerful algorithm for solving this problem is the well-known proximal point algorithm (PPA), which generates, for any initial guess, $x_0 \in H$, an iterative sequence as

$$x_{n+1} = J_{c_n}(x_n + e_n), \quad (1.1)$$

where (c_n) is a positive real sequence and (e_n) is the error sequence (see [1]). To guarantee the convergence of PPA, there are two kinds of accuracy criterion posed on the error sequence:

$$(I) \quad \|e_n\| \leq \varepsilon_n, \quad \sum_{n=0}^{\infty} \varepsilon_n < \infty \quad \text{or}$$

$$(II) \quad \|e_n\| \leq \eta_n \|\tilde{x}_n - x_n\|, \quad \sum_{n=0}^{\infty} \eta_n < \infty,$$

where $\tilde{x}_n = J_{c_n}(x_n + e_n)$. In 2001, Han and He [2] proved that in finite dimensional Hilbert space criterion (II) can be replaced by

$$(II') \quad \|e_n\| \leq \eta_n \|\tilde{x}_n - x_n\|, \quad \sum_{n=0}^{\infty} \eta_n^2 < \infty.$$

The infinite version was obtained by Marino and Xu [3].

There are various generations or modifications on the PPA. Among them Eckstein and Bertsekas [4] proposed the relaxed proximal point algorithm (RPPA):

$$x_{n+1} = (1 - \rho_n)x_n + \rho_n J_{c_n}(x_n) + e_n, \quad (1.2)$$

where $(\rho_n) \subset (0, 2)$ is a relaxation factor. The weak convergence of (1.2) is guaranteed provided that (e_n) satisfies criterion (I),

$$c_n \geq \bar{c} > 0, \quad 0 < \delta \leq \rho_n \leq 2 - \delta. \quad (1.3)$$

On the other hand, since the PPA does not necessarily converge strongly (see [5]), many authors have conducted worthwhile studies on modifying the PPA so that the strong convergence is guaranteed (see, for instance, [6-8]). In particular, Marino and Xu [3] proposed the contraction-proximal point algorithm (CPPA):

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)J_{c_n}(x_n) + e_n, \quad (1.4)$$

where the parameters above satisfy (i) $\lim_n \lambda_n = 0$, $\sum_n \lambda_n = \infty$; (ii) either $\sum_n |\lambda_{n+1} - \lambda_n| < \infty$; or $\lim_n \lambda_n / \lambda_{n+1} = 1$; (iii) $0 < \underline{c} \leq c_n \leq \bar{c} < \infty$, $\sum_n |c_{n+1} - c_n| < \infty$; (iv) $\sum_n \|e_n\| < \infty$. Under these assumptions, the CPPA converges strongly to $P_S(u)$, the projection of u onto S .

In this article, we shall focus on the RPPA and CPPA. We note that the resolvent is in fact the arithmetic mean of the identity and a nonexpansive operator. By using this fact, we relax or remove some sufficient conditions to guarantee the convergence of the algorithms. As a result, we extend and improve some recent results on the PPA.

2. Some lemmas

We know that an operator $T : H \rightarrow H$ is called (i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in H$; and (ii) firmly nonexpansive if $\langle Tx - Ty, x - y \rangle \leq \|Tx - Ty\|^2 \quad \forall x, y \in H$. Denote by $\text{Fix}(T) = \{x \in H : x = Tx\}$ the fixed point set of T . It is well known that firmly nonexpansive operators have the following properties.

Lemma 1 (Goebel-Kirk [9]). *Let T be firmly nonexpansive. Then (1) $2T - I$ is nonexpansive; (2) $\langle Tx - x, Tx - z \rangle \leq 0$ for all $x \in H$ and for all $z \in \text{Fix}(T)$.*

It is well known that J_c is firmly nonexpansive and consequently nonexpansive; moreover, $S = \text{Fix}(J_c)$. Since the fixed point set of nonexpansive operators is closed convex, the projection P_S onto the solution set S is well defined whenever $S \neq \emptyset$. Hereafter, we assume that S is nonempty. The following lemmas play an important role in our convergence analysis.

Lemma 2 (resolvent identity [3]). *Let $c, t > 0$. Then for any $x \in H$,*

$$J_c x = J_t \left(\frac{t}{c} x + \left(1 - \frac{t}{c} \right) J_c x \right).$$

Lemma 3 ([10]). *Let (ρ_n) be real sequence satisfying*

$$0 < \liminf_{n \rightarrow \infty} \rho_n \leq \limsup_{n \rightarrow \infty} \rho_n < 1.$$

Assume that (x_n) and (y_n) are bounded sequences in H satisfying $x_{n+1} = (1 - \rho_n)x_n + \rho_n y_n$. If

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0,$$

then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 4 For $r, s, > 0$, let $T_r = 2J_r - I$. Then for any $x \in H$,

$$\|T_s x - T_r x\| \leq \left| 1 - \frac{s}{r} \right| \|x - T_r x\|. \quad (2.1)$$

Proof. Using the resolvent identity, we have

$$\begin{aligned} \|T_s x - T_r x\| &= 2 \left\| J_s x - J_s \left(\frac{s}{r} x + \left(1 - \frac{s}{r} \right) J_r x \right) \right\| \\ &\leq 2 \left\| x - \left(\frac{s}{r} x + \left(1 - \frac{s}{r} \right) J_r x \right) \right\| \\ &= 2 \left| 1 - \frac{s}{r} \right| \|x - J_r x\| \\ &= \left| 1 - \frac{s}{r} \right| \|x - T_r x\|, \end{aligned}$$

where the inequality uses the nonexpansive property of the resolvent.

Lemma 5 ([11]). Let (ε_n) and (s_n) be positive real sequences. Assume that $\sum_n \varepsilon_n < \infty$. If either (i) $s_{n+1} \leq (1 + \varepsilon_n)s_n$ or (ii) $s_{n+1} \leq \varepsilon_n$ then the limit of (s_n) exists.

3. The relaxed proximal point algorithm

Under criterion (II'), Ceng et al. [12] considered another type, RPPA:

$$\begin{cases} \tilde{x}_n = J_{c_n}(x_n + e_n), \\ x_{n+1} = (1 - \rho_n)x_n + \rho_n \tilde{x}_n, \end{cases} \quad (3.1)$$

and proved the weak convergence of (3.1) under the assumptions:

$$c_n \geq \bar{c} > 0, \quad 0 < \delta \leq \rho_n \leq 1.$$

We note that the choice of (ρ_n) excludes the case whenever $\rho_n \in (1, 2)$, the overrelaxation. The overrelaxation, however, may indeed speed up the convergence of the algorithm (see [13]). Below, we shall improve their conditions on the relaxation factor from $0 < \delta \leq \rho_n \leq 1$ to $0 < \delta \leq \rho_n \leq 2 - \delta$.

Theorem 6. Assume that the following conditions hold:

- (a) $c_n \geq \bar{c} > 0$;
- (b) $0 < \delta \leq \rho_n \leq 2 - \delta$;
- (c) $\sum_n \|e_n\| \leq \eta_n \|\tilde{x}_n - x_n\|$, $\sum_n \eta_n^2 < \infty$.

Then the sequence generated by (3.1) converges weakly to a point in S .

Proof. The key point of our proof is to show $\lim_n s_n = 0$, where $s_n = \|x_n - J_{c_n}(x_n)\|$. To see this, let $z \in S$ be fixed. Since J_{c_n} is firmly nonexpansive and $z \in \text{Fix}(J_{c_n})$, applying Lemma 1 yields $\langle \tilde{x}_n - z, \tilde{x}_n - x_n - e_n \rangle \leq 0$. This together with (3.1) enables us to get

$$\begin{aligned} \|x_{n+1} - z\|^2 - \|x_n - z\|^2 &= \|(x_n - z) + \rho_n(\tilde{x}_n - x_n)\|^2 - \|x_n - z\|^2 \\ &= 2\rho_n \langle x_n - z, \tilde{x}_n - x_n \rangle + \rho_n^2 \|\tilde{x}_n - x_n\|^2 \\ &= 2\rho_n \langle \tilde{x}_n - z, \tilde{x}_n - x_n \rangle - \rho_n(2 - \rho_n) \|\tilde{x}_n - x_n\|^2 \\ &\leq 2\rho_n \langle \tilde{x}_n - z, e_n \rangle - \rho_n(2 - \rho_n) \|\tilde{x}_n - x_n\|^2 \\ &= 2\rho_n \langle \tilde{x}_n - x_n, e_n \rangle + 2\rho_n \langle x_n - z, e_n \rangle - \rho_n(2 - \rho_n) \|\tilde{x}_n - x_n\|^2 \\ &\leq 2\rho_n \|e_n\| \|\tilde{x}_n - x_n\| + 2\rho_n \|e_n\| \|x_n - z\| - \rho_n(2 - \rho_n) \|\tilde{x}_n - x_n\|^2 \\ &\leq 2\rho_n \eta_n \|\tilde{x}_n - x_n\|^2 + 2\rho_n \eta_n \|\tilde{x}_n - x_n\| \|x_n - z\| \\ &\quad - \rho_n(2 - \rho_n) \|\tilde{x}_n - x_n\|^2. \end{aligned}$$

Using the basic inequality $2ab \leq a^2 / \varepsilon + \varepsilon b^2$ ($a, b \in \mathbb{R}, \varepsilon > 0$), we arrive at

$$\begin{aligned} 2\rho_n \eta_n \|x_n - z\| \|\tilde{x}_n - x_n\| &\leq \frac{2\rho_n}{2 - \rho_n} (\eta_n \|x_n - z\|)^2 + \frac{2 - \rho_n}{2\rho_n} (\rho_n \|\tilde{x}_n - x_n\|)^2 \\ &= \frac{2\rho_n \eta_n^2}{2 - \rho_n} \|x_n - z\|^2 + \frac{\rho_n(2 - \rho_n)}{2} \|\tilde{x}_n - x_n\|^2 \\ &\leq \frac{2(2 - \delta)\eta_n^2}{\delta} \|x_n - z\|^2 + \frac{\rho_n(2 - \rho_n)}{2} \|\tilde{x}_n - x_n\|^2 \\ &= \varepsilon_n \|x_n - z\|^2 + \frac{\rho_n(2 - \rho_n)}{2} \|\tilde{x}_n - x_n\|^2, \end{aligned}$$

where $\varepsilon_n = 2(2 - \delta)\eta_n^2/\delta$ is a summable sequence. Substituting this into above yields

$$\|x_{n+1} - z\|^2 \leq (1 + \varepsilon_n) \|x_n - z\|^2 - \frac{\rho_n(2 - \rho_n - 4\eta_n)}{2} \|\tilde{x}_n - x_n\|^2.$$

Since by Lemma 5 the limit of $\|x_n - z\|^2$ exists and $\liminf_n \rho_n(2 - \rho_n - 4\eta_n) \geq \delta(2 - \delta)$, this implies that $\|\tilde{x}_n - x_n\| \rightarrow 0$. On the other hand, we note that for all $n \in \mathbb{N}$

$$s_n \leq (1 + \eta_n) \|x_n - \tilde{x}_n\| \rightarrow 0;$$

therefore, $\lim_n s_n = 0$. The rest proof is similar to that of [12, Theorem 3.1].

We now turn to the RPPA (1.2). Under the criterion (I), the assumptions on relaxation factors can be relaxed to $\sum \rho_n(2 - \rho_n) = \infty$ (see [3, Theorem 3.3]). Since the proof there is very technical, we want to restate this result with a simple proof.

Theorem 7. Assume that the following conditions hold:

- (a) $\sum_n \|e_n\| < \infty$;
- (b) $\sum_n \rho_n(2 - \rho_n) = \infty$;
- (c) $0 < \bar{c} \leq c_n \leq \tilde{c} < \infty$;
- (d) $\sum_n |c_{n+1} - c_n| < \infty$.

Then the sequence generated by (1.2) converges weakly to a point in S .

Proof. The key step is to show $\lim_n s_n = 0$, where $s_n = \|x_n - J_{c_n}(x_n)\|$. It has been shown that $\sum_n \rho_n(2 - \rho_n)s_n < \infty$ (see [3, Lemma 3.2]). Therefore, it remains to show that $\lim_n s_n$ exists. By letting $T_n = 2J_n - I$, we rewrite (2) as

$$x_{n+1} = \left(1 - \frac{\rho_n}{2}\right)x_n + \frac{\rho_n}{2}T_n x_n + e_n.$$

In view of Lemma 4 and condition (c),

$$\begin{aligned} \|T_{n+1}x_{n+1} - T_n x_n\| &\leq \|T_{n+1}x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|T_{n+1}x_n - T_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \left|1 - \frac{c_{n+1}}{c_n}\right| \|T_n x_n - x_n\| \\ &\leq \|x_n - x_{n+1}\| + \frac{|c_{n+1} - c_n|}{\bar{c}} \|T_n x_n - x_n\| \\ &\leq \|x_n - x_{n+1}\| + M|c_{n+1} - c_n|, \end{aligned}$$

where $M > 0$ is a suitable number. Consequently,

$$\begin{aligned}\|x_{n+1} - T_{n+1}x_{n+1}\| &= \left\| \left(1 - \frac{\rho_n}{2}\right)x_n + \frac{\rho_n}{2}T_nx_n + e_n - T_{n+1}x_{n+1} \right\| \\ &= \left\| \left(1 - \frac{\rho_n}{2}\right)(x_n - T_nx_n) + (T_nx_n - T_{n+1}x_{n+1}) + e_n \right\| \\ &\leq \left(1 - \frac{\rho_n}{2}\right)\|x_n - T_nx_n\| + \|T_nx_n - T_{n+1}x_{n+1}\| + \|e_n\| \\ &\leq \left(1 - \frac{\rho_n}{2}\right)\|x_n - T_nx_n\| + \|x_n - x_{n+1}\| \\ &\quad + M|c_{n+1} - c_n| + \|e_n\| \\ &= \left(1 - \frac{\rho_n}{2}\right)\|x_n - T_nx_n\| + \left\| \frac{\rho_n}{2}(x_n - T_nx_n) + e_n \right\| \\ &\quad + M|c_{n+1} - c_n| + \|e_n\| \\ &\leq \|x_n - T_nx_n\| + M|c_{n+1} - c_n| + 2\|e_n\|.\end{aligned}$$

Using $s_n = \|x_n - T_nx_n\|/2$, we therefore arrive at

$$s_{n+1} \leq s_n + \sigma_n,$$

where $\sigma_n = 2M|c_{n+1} - c_n| + 4\|e_n\|$ satisfying $\sum_n \sigma_n < \infty$ (due to (a) and (d)). By Lemma 5, we finally conclude that $\lim_n s_n = 0$.

4. The contraction-proximal point algorithm

Recently, Yao and Noor [14] extended the CPPA to the following form:

$$x_{n+1} = \lambda_n u + r_n x_n + \delta_n J_{c_n}(x_n) + e_n, \quad (4.1)$$

where $(\lambda_n), (r_n), (\delta_n) \subseteq (0, 1)$ and $\lambda_n + r_n + \delta_n = 1$. They proved the strong convergence of the algorithm provided that (i) $c_n \geq \bar{c} > 0$, $\lim_n |c_{n+1} - c_n| = 0$; (ii) $0 < \liminf_n r_n \leq \limsup_n r_n < 1$; and (iii) $\sum_n \|e_n\| < \infty$. Also, they claimed that their algorithm includes the CPPA as a special case. This is, however, not the case, because condition (ii) excludes the special case $r_n \equiv 0$. To overcome this drawback, we shall show the same result by replacing condition (ii) with the weak condition:

$$\limsup_{n \rightarrow \infty} r_n < 1 \Leftrightarrow \liminf_{n \rightarrow \infty} \delta_n > 0.$$

In this situation, the CPPA is evidently a special case of algorithm (4.1). The idea of the following proof is followed by the second author [15].

Theorem 8. *Let be (λ_n) , (r_n) and (δ_n) be parameters in (4.1). Assume that the following conditions hold:*

- (a) $\lim_n \lambda_n = 0$, $\sum_n \lambda_n = \infty$;
- (b) $\limsup_n r_n < 1 \Leftrightarrow \liminf_n \delta_n > 0$;
- (c) $c_n \geq \bar{c} > 0$, $|c_{n+1} - c_n| \rightarrow 0$;
- (d) $\sum_n \|e_n\| < \infty$.

Then the sequence generated by (4.1) converges strongly to $P_S(u)$.

Proof. All we need to do is to prove $\|x_{n+1} - x_n\| \rightarrow 0$, since the rest proof is similar to that of [14, Theorem 3.3]. To this end, set $J_n = J_{c_n}$ and $T_n = 2J_n - I$. It then follows from (4.1) that

$$\begin{aligned}x_{n+1} &= \lambda_n u + r_n x_n + \frac{\delta_n}{2}(I + T_n)x_n + e_n \\ &= \left(r_n + \frac{\delta_n}{2}\right)x_n + \lambda_n u + \frac{\delta_n}{2}T_nx_n + e_n.\end{aligned}$$

Let $\rho_n = \lambda_n + (\delta_n/2)$. Then the algorithm has the form:

$$x_{n+1} = (1 - \rho_n)x_n + \rho_n y_n, \quad (4.2)$$

where $y_n = (2\lambda_n u + \delta_n T_n x_n + 2e_n)/2\rho_n$. Using nonexpansiveness of T_n and Lemma 4, we have

$$\begin{aligned} \|T_{n+1}x_{n+1} - T_n x_n\| &\leq \|T_{n+1}x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_n x_n\| \\ &\leq \|x_{n+1} - x_n\| + \left|1 - \frac{c_{n+1}}{c_n}\right| \|T_n x_n - x_n\| \\ &\leq \|x_{n+1} - x_n\| + \frac{|c_n - c_{n+1}|}{\bar{c}} \|T_n x_n - x_n\|. \end{aligned} \quad (4.3)$$

On the other hand, it follows from the definition of y_n that

$$\begin{aligned} \|y_{n+1} - y_n\| &= \left\| \frac{1}{2\rho_{n+1}}(2\lambda_{n+1}u + \delta_{n+1}T_{n+1}x_{n+1} + 2e_{n+1}) \right. \\ &\quad \left. - \frac{1}{2\rho_n}(2\lambda_n u + \delta_n T_n x_n + 2e_n) \right\| \\ &\leq \left| \frac{\lambda_{n+1}}{\rho_{n+1}} - \frac{\lambda_n}{\rho_n} \right| \|u\| + \frac{\|e_{n+1}\|}{\rho_{n+1}} + \frac{\|e_n\|}{\rho_n} \\ &\quad + \left\| \frac{\delta_{n+1}}{2\rho_{n+1}}T_{n+1}x_{n+1} - \frac{\delta_n}{2\rho_n}T_n x_n \right\| \\ &\leq \left| \frac{\lambda_{n+1}}{\rho_{n+1}} - \frac{\lambda_n}{\rho_n} \right| \|u\| + \frac{\|e_{n+1}\|}{\rho_{n+1}} + \frac{\|e_n\|}{\rho_n} \\ &\quad + \left| \frac{\delta_{n+1}}{2\rho_{n+1}} - \frac{\delta_n}{2\rho_n} \right| \|T_{n+1}x_{n+1}\| \\ &\quad + \frac{\delta_n}{2\rho_n} \|T_{n+1}x_{n+1} - T_n x_n\|. \end{aligned} \quad (4.4)$$

Since (x_n) is bounded and T_n is nonexpansive, we can find $M > 0$ so that $(\|T_n x_n\| + \|x_n\| + \|u\|) \leq M$ for all $n \in \mathbb{N}$. Adding (4.3) and (4.4) and noting $\delta_n \leq 2\rho_n$ yield

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \left| \frac{\lambda_{n+1}}{\rho_{n+1}} - \frac{\lambda_n}{\rho_n} \right| \|u\| + \frac{\|e_{n+1}\|}{\rho_{n+1}} + \frac{\|e_n\|}{\rho_n} \\ &\quad + \left| \frac{\delta_{n+1}}{2\rho_{n+1}} - \frac{\delta_n}{2\rho_n} \right| \|T_{n+1}x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \frac{|c_n - c_{n+1}|}{\bar{c}} \|T_n x_n - x_n\| \\ &\leq \|x_{n+1} - x_n\| + M \left(\left| \frac{\lambda_{n+1}}{\rho_{n+1}} - \frac{\lambda_n}{\rho_n} \right| + \frac{\|e_{n+1}\|}{\rho_{n+1}} \right. \\ &\quad \left. + \frac{\|e_n\|}{\rho_n} + \left| \frac{\delta_{n+1}}{2\rho_{n+1}} - \frac{\delta_n}{2\rho_n} \right| + \frac{|c_n - c_{n+1}|}{\bar{c}} \right). \end{aligned}$$

With the knowledge that $\|e_n\| \rightarrow 0$ and

$$\frac{\lambda_n}{\rho_n} = \frac{2\lambda_n}{2\lambda_n + \delta_n} \rightarrow 0, \quad \frac{\delta_n}{2\rho_n} = \frac{\delta_n}{2\lambda_n + \delta_n} \rightarrow 1,$$

we therefore deduce from (b) and (c) that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \\ & \leq \limsup_{n \rightarrow \infty} M \left(\left| \frac{\lambda_{n+1}}{\rho_{n+1}} - \frac{\lambda_n}{\rho_n} \right| + \frac{\|e_{n+1}\|}{\rho_{n+1}} + \frac{\|e_n\|}{\rho_n} \right. \\ & \quad \left. + \left| \frac{\delta_{n+1}}{2\rho_{n+1}} - \frac{\delta_n}{2\rho_n} \right| + \frac{|c_n - c_{n+1}|}{\bar{c}} \right) \rightarrow 0. \end{aligned}$$

Note that $\liminf_n \rho_n = \liminf_n (\delta_n/2) > 0$ and $\limsup_n \rho_n = \limsup_n (\delta_n/2) \leq 1/2 < 1$. On the other hand, it is easy to check that (x_n) is bounded and so is (y_n) . We therefore apply Lemma 3 to yield $\lim_n \|x_n - y_n\| = 0$. By means of (4.2), we finally have

$$\|x_{n+1} - x_n\| = \rho_n \|x_n - y_n\| \rightarrow 0,$$

and thus the required result at once follows.

As a corollary, we improve [3, Theorem 4.1] as follows.

Theorem 9. *Assume that the following conditions hold:*

- (a) $\lim_n \lambda_n = 0$, $\sum_n \lambda_n = \infty$;
- (b) $c_n \geq \bar{c} > 0$, $|c_{n+1} - c_n| \rightarrow 0$;
- (c) $\sum_n \|e_n\| < \infty$.

Then the sequence generated by (1.4) converges strongly to $P_S(u)$.

Abbreviations

CPPA: contraction-proximal point algorithm; PPA: proximal point algorithm; RPPA: relaxed proximal point algorithm.

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Authors' contributions

Both authors contributed equally to this work. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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